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The theta-Laguerre calculus formulation of the Li/Keiper constants

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Abstract

The Riemann hypothesis is equivalent to the nonnegativity of a sequence of real constants $\{\lambda_k\}_{k=1}^{\infty}$, that are certain logarithmic derivatives of the Riemann xi function evaluated at unity. We re-express these constants using the theta-Laguerre calculus. By using integral representations, we reformulate the coefficients $\{\lambda_k\}_{k=1}^{\infty}$ together with a closely related sequence $\{a_j\}_{j=0}^{\infty}$. We present a decomposition of the quantities a_j into superdominant and subdominant components and give an upper bound on the former and an asymptotic lower bound for the latter. Sufficient estimation of these quantities would lead to confirmation of the Riemann hypothesis.

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Keywords: Li/Keiper constants; Riemann zeta function; Riemann xi function; Logarithmic derivatives; Riemann hypothesis; Li criterion; Associated Laguerre polynomials; Theta-Laguerre calculus; Laurent expansion

1. Introduction

The Riemann hypothesis is equivalent to the Li criterion governing the sequence of real constants, $\{\lambda_k\}_{k=1}^{\infty}$, that are certain logarithmic derivatives of the Riemann xi function evaluated at unity. In 1992, Keiper effectively introduced these constants [22], while later Li completed the equivalence [24]. This equivalence results from a necessary and sufficient condition that the logarithmic derivative of the function $\xi[1/(1-z)]$ be analytic in the unit disk, where ξ is the Riemann xi function. The Li equivalence [24] states that a necessary and sufficient condition for

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the nontrivial zeros of the Riemann zeta function to lie on the critical line $\operatorname{Re} s = \frac{1}{2}$ is that $\{\lambda_k\}_{k=1}^{\infty}$ is nonnegative for every integer k.

In this paper, we investigate underlying structure of the Li/Keiper constants and find the theta-Laguerre calculus to be key. After introducing some definitions and notation, we list a series of results. An outline of their proofs is given in the succeeding section. The paper is concluded with a Discussion section.

The function ξ is determined from ζ by the relation [13,14,19,21,28,25]

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1}$$

and satisfies the functional equation $\xi(s) = \xi(1-s)$. The sequence $\{\lambda_n\}_{n=1}^{\infty}$ is defined by

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\ln\xi(s)]_{s=1}. \tag{2}$$

The λ_j's are connected to sums over the nontrivial zeros of $\zeta(s)$ by way of [22,24]

$$\lambda_n = \sum_{\rho}\left[1-\left(1-\frac{1}{\rho}\right)^n\right], \tag{3}$$

and the first is known exactly [25,7–9],

$$\lambda_1 = -\frac{\ln\pi}{2} + \frac{\gamma}{2} + 1 - \ln 2. \tag{4}$$

Since the sum over all complex zeros in Eq. (3) includes all pairs ρ and $1-\rho$, there is the symmetry $\lambda_n = \lambda_{-n}$. A set of companion sums to Eq. (3),

$$\sigma_k \equiv \sum_{\rho}\frac{1}{\rho^k}, \tag{5}$$

has been often considered in the literature. These sums are absolutely convergent for $k \geqslant 2$. The σ_k coefficients appear in an expansion of the logarithm of the xi function about $s = 1$ [22,8],

$$\ln\xi(s) = -\ln 2 - \sum_{k=1}^{\infty}(-1)^k\frac{\sigma_k}{k}(s-1)^k, \tag{6}$$

and are accordingly of high interest for a variety of reasons.

Proposition 1. (i) *For $n \geqslant 1$ an integer we have*

$$\lambda_n = \sum_{\rho}\int_0^{\infty} e^{-\rho u}L_{n-1}^1(u)\,du, \tag{7}$$

where L_n^{α} is an associated Laguerre polynomial. (ii)

$$\lambda_n = -\sum_{j=1}^{n}(-1)^j\binom{n}{j}\sigma_j, \tag{8}$$

and $\lambda_1 = \sigma_1$.

Proposition 2. *Let* $\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, *a Jacobi theta function* [1,6], *and* $\varphi(z) = \xi[1/(1-z)]$. *Set*

$$\varphi(z) = 1 + \sum_{j=1}^{\infty} a_j z^j, \tag{9}$$

with $a_0 \equiv 1$ [24]. *Then we have the representations*

$$a_j = \frac{2}{j}\int_1^{\infty} [x^{3/2}\psi'(x)]' \left[L_{j-1}^1\left(-\frac{\ln x}{2}\right) - \frac{1}{\sqrt{x}} L_{j-1}^1\left(\frac{\ln x}{2}\right)\right] \ln x \, dx, \quad j \geqslant 1, \tag{10a}$$

$$\begin{aligned} a_j = -\frac{1}{j}\int_1^{\infty} \psi'(x) &\left\{\left[\sqrt{x} L_{j-2}^2\left(-\frac{\ln x}{2}\right) + L_{j-1}^2\left(\frac{\ln x}{2}\right)\right] \ln x \right. \\ &\left. +2\left[\sqrt{x} L_{j-1}^1\left(-\frac{\ln x}{2}\right) - L_{j-1}^1\left(\frac{\ln x}{2}\right)\right]\right\} dx, \quad j \geqslant 1, \end{aligned} \tag{10b}$$

$$\begin{aligned} a_j = \frac{1}{j}\int_1^{\infty} \frac{\psi(x)}{x^{1/2}} &\left\{\frac{\ln x}{2}\left[L_{j-2}^3\left(-\frac{\ln x}{2}\right) - \frac{1}{x^{1/2}} L_{j-2}^3\left(\frac{\ln x}{2}\right)\right]\right. \\ &+ L_{j-1}^2\left(-\frac{\ln x}{2}\right) + \frac{1}{x^{1/2}} L_{j-1}^2\left(\frac{\ln x}{2}\right) + L_{j-2}^2\left(-\frac{\ln x}{2}\right) \\ &\left. + \frac{1}{x^{1/2}} L_{j-2}^2\left(\frac{\ln x}{2}\right)\right\} dx, \quad j \geqslant 1. \end{aligned} \tag{10c}$$

Proposition 3. *We have for complex* z (i) *the classical theta-based representation* (*e.g.*, [14,28])

$$\varphi(z) = 4\int_1^{\infty} [x^{3/2}\psi'(x)]' [x^{-1/2} x^{1/2(1-z)} + x^{-1/2(1-z)}] \, dx, \quad |z| < 1, \tag{11}$$

and (ii) *the recurrence relation* [24]

$$\lambda_n = n a_n - \sum_{j=1}^{n-1} \lambda_j a_{n-j}, \quad n \geqslant 1. \tag{12}$$

Proposition 4. *Let* $a_j = \mathcal{M}_j + m_j$, *where*

$$\mathcal{M}_j = \frac{2}{j}\int_1^{\infty} [x^{3/2}\psi'(x)]' L_{j-1}^1\left(-\frac{\ln x}{2}\right) \ln x \, dx, \quad j \geqslant 1, \tag{13a}$$

$$m_j = -\frac{2}{j}\int_1^{\infty} [x^{3/2}\psi'(x)]' \frac{1}{\sqrt{x}} L_{j-1}^1\left(\frac{\ln x}{2}\right) \ln x \, dx, \quad j \geqslant 1. \tag{13b}$$

Then $|m_j| \ll |\mathcal{M}_j|$ *for* $j \gg 1$ *and for* $j \to \infty$ *we have*

$$\begin{aligned} \mathcal{M}_j \gtrsim \frac{j}{\ln^{1/2}\sqrt{j}} e^{(2-\ln 2)\sqrt{j}+O(1)} &\left[\frac{1}{\pi} - \frac{2}{3\pi\sqrt{j}} + O\left(\frac{1}{j}\right)\right] \left[\ln\frac{\sqrt{j}}{\pi} - \ln\left(\ln\frac{\sqrt{j}}{\pi}\right)\right]^{\sqrt{j}-2} \\ \times \exp&\left[-\frac{(\sqrt{j}-5/2)}{\ln(\sqrt{j}-5/2)}\right], \quad j \to \infty. \end{aligned} \tag{14}$$

2. Proofs of propositions

Proof of Proposition 1. (i) We give two different proofs. *Method* 1: We first use integration by parts with $L_n'(x) = -L_{n-1}^1(x)$:

$$\sum_\rho \int_0^\infty e^{-\rho u} L_{n-1}^1(u)\,du = -\sum_\rho \int_0^\infty e^{-\rho u} \frac{d}{du} L_n(u)\,du = \sum_\rho \left[1 - \rho \int_0^\infty e^{-\rho u} L_n(u)\,du\right],$$

where $L_n(0) = 1$ was used. It is known that all complex zeros of $\zeta(s)$ lie in the critical strip, so that $\mathrm{Re}\,\rho > 0$. Then by the Laplace transform of a Laguerre polynomial [17, p. 844], $\int_0^\infty e^{-bx} L_n(x)\,dx = (b-1)^n b^{-n-1}$, for $\mathrm{Re}\,b > 0$, Eq. (7) is seen to be equivalent to Eq. (3). Since $L_0^1(x) = 1$, we recover the relation $\lambda_1 = \sigma_1$.

Method 2: From Eq. (7) we may write

$$\lambda_n = \sum_\rho \frac{1}{\rho} \int_0^\infty e^{-v} L_{n-1}^1\left(\frac{v}{\rho}\right) dv, \tag{15}$$

and it is known that [3,15]

$$L_m^\beta(\tau x) = \sum_{n=0}^m \binom{\beta+m}{m-n} \tau^n (1-\tau)^{m-n} L_n^\beta(x). \tag{16}$$

This property permits the recovery of the defining sum-over-complex-zeros relation (3) of the Li/Keiper constants:

$$\lambda_n = \sum_\rho \sum_{j=1}^n \binom{n}{j} \frac{1}{\rho^j} \left(1 - \frac{1}{\rho}\right)^{n-j} \left[-\int_0^\infty e^{-v} \frac{d}{dv} L_j(v)\,dv\right]. \tag{17}$$

Upon integration by parts in the brackets on the right side of this equation, the orthogonality relation $\int_0^\infty \exp(-v) L_j(v)\,dv = 0$ for $j \geqslant 1$ determines that this factor is simply unity. Then

$$\lambda_n = \sum_\rho \sum_{j=1}^n \binom{n}{j} \frac{1}{\rho^j} \left(1 - \frac{1}{\rho}\right)^{n-j} = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]. \tag{18}$$

(ii) We may integrate by parts alternatively with respect to the exponential factor in Eq. (7), obtaining for instance, $\lambda_n = n\lambda_1 - \sum_\rho \frac{1}{\rho} \int_0^\infty L_{n-2}^2(u) e^{-\rho u}\,du$. We continue this process, integrating by parts n times in Eq. (7). By using $L_{n-j}^j(0) = \binom{n}{n-j} = \binom{n}{j}$ we recover Eq. (8). □

Remark. Eq. (8) may be obtained from Eq. (3) simply by binomial expansion. We have shown an alternative based upon the properties of the associated Laguerre polynomial.

Proof of Proposition 2. The forms (10b) and (10c) follow from (10a) by using integration by parts and the recursion relation $L_{j-1}^\alpha(z) + L_{j-2}^{\alpha+1}(z) = L_{j-1}^{\alpha+1}(z)$ for $\alpha = 1$ and 2.

Eq. (10a) follows by re-expressing a_j of Ref. [24] in terms of the confluent hypergeometric function ${}_1F_1$ and then transforming. In fact, by comparing with expressions given in Ref. [7]

or otherwise, we may write

$$a_j = 2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(j+1)(j+2)\cdots(j+n)}{n!(n+1)!}\xi^{(n+1)}(1). \tag{19}$$

The relation $a_j = (1/j!)(d^j/dz^j)\varphi(z)|_{z=0}$ given in this equation is a result of the Faá di Bruno formula for the derivatives of a composite function. We have

$$a_j = 2\int_1^{\infty}[x^{3/2}\psi'(x)]'\ln x\left[\frac{1}{\sqrt{x}}{}_1F_1\left(j+1, 2, \frac{\ln x}{2}\right) - {}_1F_1\left(j+1, 2, -\frac{\ln x}{2}\right)\right]dx. \tag{20}$$

From Kummer's first transformation [2] we have ${}_1F_1(j+1, 2, x) = e^x L_{j-1}^1(-x)/j$ and Eq. (10a) follows. □

Remark. We recall that

$$n\,{}_1F_1(1-n, 2, x) = L_{n-1}^1(x) = \frac{(-x)^{n-1}}{(n-1)!}{}_2F_0\left(1-n, -n; .; -\frac{1}{x}\right),$$

where the latter form follows by reordering of the series for the hypergeometric function ${}_pF_q$.

Proof of Proposition 3. (i) Eq. (11) is obtained from Eq. (9) by applying

Lemma 1.

$$\sum_{n=1}^{\infty}\frac{L_{n-1}^1(x)}{n}w^n = \frac{1}{x}[1 - e^{xw/(w-1)}], \quad |w| < 1. \tag{21}$$

Eq. (21) is obtained by integrating the generating function equation [2] of the associated Laguerre polynomial $L_n^1(x)$.

(ii) Eq. (12) is obtained from Eqs. (7) and (10) by applying

Lemma 2.

$$\sum_{j=1}^{n-1}\frac{L_{j-1}^1(x)L_{n-j-1}^1(y)}{n-j} = \frac{1}{y}\left[L_{n-1}^1(x) - L_{n-1}^1(x+y)\right]. \tag{22}$$

This result is obtained from the generating function equation of L_n^1 together with the previous result (21). □

Remark. The extension of Lemmas 1 and 2 to general polynomials L_n^α has very recently been presented in Ref. [11].

Proof of Proposition 4. The splitting $a_j = \mathcal{M}_j + m_j$ follows specifically from Eq. (10a). In Eq. (13a), each factor of the integrand is positive for $x > 1$ and therefore necessarily $\mathcal{M}_j > 0$ for each j.

From the series definition of the associated Laguerre polynomials, it is obvious that $|L_n^\alpha(x)| \leqslant L_n^\alpha(-x)$ for all $x \geqslant 0$ and $\alpha > -1$. A separate estimation yields the bound $|m_j| \leqslant 1/2j^{3/4}$.

Remark. This bound is conservative and improvable. The decreasing sequence m_j oscillates about zero. The increase of a_j with j is given by the $\mathcal{M}_j$ component.

A lower bound for $\mathcal{M}_j$ can be developed in the following way. Each term in Eq. (13a) coming from the successive terms of the associated Laguerre polynomial is positive. In seeking a nontrivial lower bound, we can consider the term of this polynomial with largest coefficient. If we consider the interpolating coefficient $\Gamma(n+1)/\Gamma(m+1)\Gamma(n-m+1)\Gamma(m)$, it will be maximized for $\psi_0(n-m+1) = 2\psi_0(m) + 1/m$, where ψ_0 is the digamma function. This gives the integer closest to $\sqrt{n+\frac{3}{4}} - \frac{1}{2}$ as an approximate solution for m. In turn from Eq. (13a) this gives

$$\mathcal{M}_j \geqslant \frac{2^{5/2}\Gamma(j)}{2^{\sqrt{j+3/4}}\Gamma(\sqrt{j+3/4}+1/2)\Gamma(j-\sqrt{j+3/4}+1/2)\Gamma(\sqrt{j+3/4}-1/2)} \times \int_1^\infty [x^{3/2}\psi'(x)]'(\ln x)^{\sqrt{j+3/4}-1/2}\,dx. \tag{23}$$

From Stirling's formula, as $j \to \infty$, the prefactor in Eq. (23) is given by

$$\frac{2^{5/2}\Gamma(j)}{2^{\sqrt{j+3/4}}\Gamma(\sqrt{j+3/4}+1/2)\Gamma(j-\sqrt{j+3/4}+1/2)\Gamma(\sqrt{j+3/4}-1/2)} \sim 2^{-\sqrt{j}} e^{2\sqrt{j}-1/2-\ln 2+1/12j+O(1/j)} \left[\frac{1}{\pi} - \frac{2}{3\pi\sqrt{j}} + \frac{7}{72\pi j} - \frac{103}{810\pi j^{3/2}} + O\left(\frac{1}{j^2}\right)\right], \quad j \to \infty. \tag{24}$$

Elsewhere we have applied the method of steepest descent to a class of integrals generalizing that of Eq. (23), namely to integrals of the form $I_j(a,b) \equiv \int_1^\infty [x^a\psi'(x)]' x^b \ln^{f(j)} x\,dx$, where $f(j) > 2$ is assumed to be monotonically increasing in j, $a > 1$, and $a + b \neq 1$.

We have by integration by parts and a change of variable

$$I_j(a,b) = \int_1^\infty x^{a+b-2}\psi(x)\ln^{f(j)-2} x[(a+b-1)b\ln^2 x + f(a+2b-1)\ln x + f(f-1)]\,dx \tag{25a}$$

$$= \int_0^\infty e^{(a+b-1)y}\psi(e^y)y^{f(j)-2}[(a+b-1)by^2 + f(a+2b-1)y + f(f-1)]\,dy. \tag{25b}$$

We write $I_j = \int_0^\infty e^{g(y)}\,dy$ and $\psi(x) = e^{-\pi x}[1+\psi_2(x)] = e^{-\pi x}[1 + O(e^{-3\pi x})]$. Define y^* as the solution of the equation $dg/dy = 0$. We have

$$\frac{dg}{dy} = a+b-1+\frac{f(j)-2}{y} + \frac{2(a+b-1)by + f(a+2b-1)}{(a+b-1)by^2 + f(a+2b-1)y + f(f-1)} - \pi e^y + \frac{1}{1+\psi_2(e^y)}\frac{d}{dy}\psi_2(e^y) \tag{26a}$$

and

$$\frac{d^2g}{dy^2} = -\frac{[f(j)-2]}{y^2} - \pi e^y + \frac{d^2}{dy^2}\ln[1+\psi_2(e^y)] + \frac{-2b^2(a+b-1)^2y^2 - 2b(a+b-1)(a+2b-1)y - 2b^2(1+f) - 2(a-1)b(f+1) - f^2(a-1)^2}{[(a+b-1)by^2 + f(a+2b-1)y + f(f-1)]^2}. \tag{26b}$$

In the present special case, we have $b = 0$. We then have $dg/dy \to \infty$ as $y \to 0$, $dg/dy \to -\infty$ as $y \to \infty$, and $d^2g/dy^2 < 0$ on $(0,\infty)$. This ensures that there is a unique solution y^* of

$dg/dy = 0$. The approximate equation $[f(j) - 2]/y = \pi \exp(y)$ is asymptotically solved as $f \to \infty$ by

$$y^* \sim \left[\ln \left(\frac{f(j) - 2}{\pi} \right) - \ln \ln \left(\frac{f(j) - 2}{\pi} \right) \right]. \tag{27}$$

To leading order in large f we have $|d^2g/dy^2|_{y=y^*} = (f-2)(1+1/y^*)/y^*$.

We then find

$$\begin{aligned} I_j(a,b) &\sim \frac{\sqrt{2\pi} \exp g(y^*)}{\left| \frac{d^2 g}{dy^2}(y^*) \right|^{1/2}} \\ &\sim \frac{f(f-1)(f-2)^{a+b-3/2}}{\ln^{a+b-1}(f-2)} \left[\ln \left(\frac{f-2}{\pi} \right) - \ln \left(\ln \left(\frac{f-2}{\pi} \right) \right) \right]^{f-3/2} \\ &\quad \times \exp \left[-\frac{(f-2)}{\ln(f-2)} \right]. \end{aligned} \tag{28}$$

By applying this result, the integral in Eq. (23) is given asymptotically as $j \to \infty$ by

$$\begin{aligned} \int_1^\infty [x^{3/2}\psi'(x)]'(\ln x)^{\sqrt{j+3/4}-1/2}\, dx &\sim \frac{j}{\ln^{1/2} \sqrt{j}} \left[\ln \frac{\sqrt{j}}{\pi} - \ln \left(\ln \frac{\sqrt{j}}{\pi} \right) \right]^{\sqrt{j}-2} \\ &\quad \times \exp \left[-\frac{(\sqrt{j} - 5/2)}{\ln(\sqrt{j} - 5/2)} \right], \quad j \to \infty. \end{aligned} \tag{29}$$

In writing Eq. (29), we have used the asymptotic form $\sqrt{j + \frac{3}{4}} - a \sim \sqrt{j}$, for $a = \frac{1}{2}, \frac{3}{2}$, and $\frac{5}{2}$ as $j \to \infty$ in the leading multiplicative factors, and similarly within the argument of the logarithm function. Therefore, by combining Eqs. (24) and (29) we obtain Eq. (14). $\square$

Remark. Eq. (14), together with the right side of Eq. (12), shows that the Li/Keiper constants arise as the difference of two very rapidly increasing terms.

3. Discussion

In our research program of the last few years, we have shown among other results that the leading order of the Li/Keiper constants λ_n is $O(n \ln n)$, effectively verifying a conjecture of Keiper [7–9]. What remains for a rigorous confirmation of the Riemann hypothesis is a verification that the subdominant contribution of the Li/Keiper constants is at most linear in n. In fact, we have conjectured that aside from explicitly linear terms in λ_n, the remaining terms are $O(n^{1/2+\varepsilon})$ where $\varepsilon > 0$ is otherwise arbitrary [8–10]. Numerical computations to date are in agreement with this conjecture [27].

In this paper, we investigated the underlying analytic structure of the Li/Keiper constants. By using various integral representations, we reformulated these constants together with a closely related sequence $\{a_j\}_{j=0}^\infty$. We described a decomposition of the quantities a_j as $a_j = m_j + \mathcal{M}_j$ with $|a_j| \ll \mathcal{M}_j$ and described how sufficient estimation of them would lead to confirmation of the Riemann hypothesis. For instance, improved characterization of a_j would allow refined estimation of λ_k from the recursion relation (12). This research opens several new possibilities for understanding the underlying algebraic structure of the a_j and λ_k coefficients.

From improved numerical calculation to height $T \simeq 2.38 \times 10^{12}$ [16], it is now known that at least the first 10 trillion complex zeros of the zeta function lie on the critical line. For our purposes, this effectively ensures that approximately the first 10^{26} λ_k's are nonnegative, and this fact may have significant implications for our investigations, indicating, for instance, that working asymptotically in k will suffice.

Eq. (7) tells us that information on functions $\mathcal{V}_f(u) = \sum_\rho \exp(-\rho u) f(u)$, $u > 0$, where f is included to ensure convergence, could give information on the Li/Keiper constants and vice versa. Indeed, Eq. (8) may be formally inverted to yield $\mathcal{V}_1$ as an expansion in associated Laguerre polynomials. By relating the functions $\mathcal{V}_f$ to the V function of Cramér [12], $V(t) = \sum_{\mathrm{Im}\,\rho>0} \exp(\rho t)$, $\mathrm{Im}\, t > 0$, direct progress in this direction may be possible. Cramér's V function is known to satisfy a functional equation and some of its asymptotic relations are also known [12,18]. In fact, when $f = 1$, $\mathcal{V}_1(u) = 2\,\mathrm{Re}\, k(-u)$, where the function k has been investigated by Kaczorowski and has a known exact asymptotic form as $u \to 0$ [20]. Under the Riemann hypothesis, $\exp(-u/2)k(u)$ is almost periodic in the L^2 sense and its limit as $u \to \infty$ does not exist [20]. However, in contrast, its real part changes sign infinitely many times [20].

It is worthwhile to examine the form of λ_j under the assumption of the Riemann hypothesis. For this we introduce the functions [23]

$$\Lambda^\alpha_{n,\mu}(x,t) \equiv \frac{L^\alpha_n(|\mu|x)}{L^\alpha_n(0)} e^{-|\mu|x/2+i\mu t}, \quad x \geqslant 0, \tag{30}$$

where μ and t are real, that have yet more properties than the associated Laguerre polynomials themselves. Then we have

$$\lambda_n = n \sum_{\alpha \in R} \int_0^\infty [\Lambda^1_{n-1,1}(u,\alpha) + \Lambda^1_{n-1,-1}(u,\alpha)]\, du \quad [\mathrm{RH}]. \tag{31}$$

The functions Λ have a *positive* convolution structure and a group theoretic interpretation as spherical functions [23]. On a certain space R^{2n+1} the convolution is commutative. Furthermore, there is the linearization formula

$$\Lambda^\alpha_{m,\mu}(x,0)\Lambda^\alpha_{n,\nu}(x,0) = \sum_{i=0}^{m+n} c_i \Lambda^\alpha_{m+n-i,\mu+\nu}(x,0), \tag{32}$$

where $c_i \geqslant 0$ if $\mu, \nu \geqslant 0$ [3,23]. These and related formulas imply further properties of the Li/Keiper constants, and these will be presented elsewhere.

The unitary group $U(n)$ acts in a canonical way on the non-Abelian nilpotent Heisenberg Lie group $H_n = C^n \times R$, and the orbit space $H_n^{U(n)}$ may be identified with a commutative hypergroup on $R \times [0, \infty)$ [26]. Given in addition a Weil formula for subgroups leading to an adjoint relation [26], one may wonder if a sort of cohomology may be developed.

The present investigation seems to offer important clues to additional algebraic structure for the Li/Keiper constants and the theta-Laguerre calculus. Besides connections with action on the Heisenberg group [5,26] there are possible connections with the Laguerre ensemble of random matrix theory (e.g., [4]). In this ensemble, random (Wishart) matrices $A = X^+X$, where X is a random $M \times N$ matrix (with $M \geqslant N$) having either real, complex, or quaternion entries that are independent and normally distributed.

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